Matrix Simplification Using Given's Rotations

A Givens plane rotation, which is the generalization of a simple rotation in $\Re^2$, has the general block form

$$
\mathbf{Q} \;=\; \begin{bmatrix}
\mathbf{I} & \vdots & \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{0} \\
\cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\
\mathbf{0} & \vdots & c & \vdots & \mathbf{0} & \vdots & -s & \vdots & \mathbf{0} \\
\cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\
\mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{I} & \vdots & \mathbf{0} & \vdots & \mathbf{0} \\
\cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\
\mathbf{0} & \vdots & s & \vdots & \mathbf{0} & \vdots & c & \vdots & \mathbf{0} \\
\cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\
\mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{I}
\end{bmatrix} \quad , \quad c^2 + s^2 = 1
$$

Note that, trivially,

$$
\mathbf{Q}^T \;=\; \begin{bmatrix}
\mathbf{I} & \vdots & \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{0} \\
\cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\
\mathbf{0} & \vdots & c & \vdots & \mathbf{0} & \vdots & s & \vdots & \mathbf{0} \\
\cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\
\mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{I} & \vdots & \mathbf{0} & \vdots & \mathbf{0} \\
\cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\
\mathbf{0} & \vdots & -s & \vdots & \mathbf{0} & \vdots & c & \vdots & \mathbf{0} \\
\cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\
\mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{I}
\end{bmatrix}
$$

and furthermore, we can easily show

$$
\mathbf{Q}\,\mathbf{Q}^T \;=\; \begin{bmatrix}
\mathbf{I} & \vdots & \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{0} \\
\cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\
\mathbf{0} & \vdots & c^2 + s^2 & \vdots & \mathbf{0} & \vdots & 0 & \vdots & \mathbf{0} \\
\cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\
\mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{I} & \vdots & \mathbf{0} & \vdots & \mathbf{0} \\
\cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\
\mathbf{0} & \vdots & 0 & \vdots & \mathbf{0} & \vdots & c^2 + s^2 & \vdots & \mathbf{0} \\
\cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\
\mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{I}
\end{bmatrix} = \mathbf{I}
$$

and therefore, by definition, Givens rotation matrices are orthogonal.

Moreover, multiplying any matrix on the left by a Givens rotation matrix produces the results:

$$\begin{bmatrix} \mathbf{I} & \vdots & \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{0} \\ \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\ \mathbf{0} & \vdots & c & \vdots & \mathbf{0} & \vdots & -s & \vdots & \mathbf{0} \\ \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\ \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{I} & \vdots & \mathbf{0} & \vdots & \mathbf{0} \\ \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\ \mathbf{0} & \vdots & s & \vdots & \mathbf{0} & \vdots & c & \vdots & \mathbf{0} \\ \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\ \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{(1:(i-1),:)} \\ \cdots\cdots\cdots \\ \mathbf{A}_{(i,:)} \\ \cdots\cdots\cdots \\ \mathbf{A}_{((i+1):(j-1),:)} \\ \cdots\cdots\cdots \\ \mathbf{A}_{(j,:)} \\ \cdots\cdots\cdots \\ \mathbf{A}_{((j+1):n,:)} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{(1:(i-1),:)} \\ \cdots\cdots\cdots \\ c\mathbf{A}_{(i,:)} - s\mathbf{A}_{(j,:)} \\ \cdots\cdots\cdots \\ \mathbf{A}_{((i+1):(j-1),:)} \\ \cdots\cdots\cdots \\ s\mathbf{A}_{(i,:)} + c\mathbf{A}_{(j,:)} \\ \cdots\cdots\cdots \\ \mathbf{A}_{((j+1):n,:)} \end{bmatrix}$$

or, in other words, a Givens rotation performs the **simultaneous** row operations:

$$\begin{aligned} R_i &\leftarrow cR_i - sR_j \\ R_j &\leftarrow sR_i + cR_j \end{aligned}$$

which is not an elementary row operation. Nevertheless, because the Givens rotation matrix is non-singular (after all, we've just shown $\mathbf{Q}^{-1} = \mathbf{Q}^T$), a multiplication of this type can be used to zero out any single selected element of another matrix without altering the solution set. For example, suppose we want to make a_{jk} equal to zero, using rows i and j of $\mathbf{A}$. Simply note that, according to the above,

$$a_{jk} \leftarrow s\, a_{ik} \ + \ c\, a_{jk}$$

and this will in fact equal zero if we choose

$$c \ = \ \frac{a_{ik}}{\sqrt{a_{ik}^2 + a_{jk}^2}} \quad , \quad s \ = \ -\frac{a_{jk}}{\sqrt{a_{ik}^2 + a_{jk}^2}}$$

Note, however, this is a fairly "expensive" way to zero out elements, since multiplying on the left by the above matrix requires about $6n$ flops, while achieving the same result with a lower triangular elementary matrix requires only about $2n$ flops.